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LIMIT THEOREMS FOR MARKOV RANDOM WALK DISCRIBED BY SUMS OF VALUES OF FIRST-ORDER AUTO REGRESSIVE PROCESSES WITH RANDOM COEFFICIENT

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Abstract. In the paper the law of large numbers and central limit theorem are proved for Markov random walk described by the sum of the values of first-other autoregressive processes with a random coefficient.

Keywords: Law of large numbers, central limit theorem, first-other autoregressive processes with, random coefficient.

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1. Introduction

Let $\xi_n, n \ge 1$ be a sequence of independent identically distributed random variables determined on the probability space (Ω, F, P).

As is known ([5],[6]) the first-order autoregressiive process with random coefficient ($(RCAR(1))$ is the solution of a recurrent equation of the form

$$X_n = \beta X_{n-1} + \alpha \xi_n, \quad n = 1,2... \tag{1}$$

where $\beta = \beta(\omega)$ and $\alpha = \alpha(\omega)$ $\omega \in \Omega,$ are some random variables.

We will assume that the initial value X_0 of the process is independent of the innovation $\{\xi_n\}$, and random variables β, α are independent between them selves and do not depend on X_0 and ξ_n for all $n \ge 1$.

Model (1.1) of $(RCAR(1))$ process arises in theoretical and also in practical problem of theory of time series [10, 11].

Note that in the case when α and β are fixed real numbers, the model of the form (1) coincides with the random coefficient first order autoregressive process of the form

$$Y_k = \theta Y_{k-1} + \eta_n, \tag{2}$$

where θ is a fixed number, and $\eta_n, n \ge 1$ are independent identically distributed random variables [1, 3-9].

Recently the theory of nonlinear renewal has ben intensively developed for Markov random walks described by the nonrandom coefficient first order autoregressive process and also with random coefficients.

Let us consider the following Markov random walk described by the $(RCAR(1))$ process of the form (1)

$$S_n = \sum_{k=0}^{n} X_k, \quad n \geq 1 \tag{3}$$

It is clear that in the case of the scheme (2) $(X_k = Y_k)$ for $\theta = 0$ the sum of the form S_n (3) forms a usual classic random walk described by the sums of independent identically distributed random variables.

The Markov random walk (3) for the model (2) has been considered in the work [5, 6] , where limit theorems were proved .

In the present work we prove a theorem on the law of large numbers and central limit theorems for the sum $S_n, n \geq 1$ for the model $(RCAR(1))$ of the form (1).

Similar limit theorems were proved in [2] for the series of Markov random walks described by the first order autoregressive process $(RCAR(1))$ with a constant (non-rondom|) coefficient.

2. Formulation and proof of the main results.

The following random theorem on the law large numbers for $S_n, n \geq 1$ is valid.

Theorem 1. Let $E|X_0| < \infty,\quad E|\xi_1| < \infty$ and $P(0 \leq \beta < 1-\varepsilon) = 1$ for some $\varepsilon \in (0,1)$ and $P(\alpha \neq 0) = 1$.

Then,

1) we have the convergence in probability $\frac{1-\beta}{\alpha}\frac{S_n}{n} \xrightarrow{P} = m = E\xi_1$ as $n \to \infty$.

2) If $\frac{X_n}{n} \xrightarrow{a \cdot s} 0,$ then $\frac{1-\beta}{\alpha}\frac{S_n}{n} \xrightarrow{a \cdot s} m, \quad n \to \infty.$

Proof. Summing over $k = \overline{1,n}$ in the equality (1) we have

$$\sum_{k=1}^{n} X_k = \beta \sum_{k=1}^{n} X_{k-1} + \alpha \sum_{k=1}^{n} \xi_k \tag{4}$$

Taking into account

$$\sum_{k=1}^{n} X_{k-1} = \sum_{k=1}^{n} X_k + \left(X_0 - X_n\right)$$

from (4) we have

$$(1-\beta)\sum_{k=1}^{n} X_k = \beta(X_0 - X_n) + \alpha\sum_{k=1}^{n}\xi_k$$

or

$$\frac{(1-\beta)}{\alpha}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{\beta}{\alpha_n}(X_0 - X_n) + \frac{1}{n}\sum_{k=1}^{n}\xi_k. \tag{5}$$

Prove that

$$\frac{X_0 - X_n}{n} \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{6}$$

It is clear that the condition $E|X_0| < \infty,$ and the Markov inequality yields

$$P(|X_0| > n\varepsilon) < \frac{E|X_0|}{n\varepsilon}$$

Hence we have:

$$\frac{X_0}{n} \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{7}$$

By means of the successive operations, from (1) we can obtain

$$X_n = \beta^n X_0 + \sum_{k=0}^{n-1} \alpha\beta^k \xi_{n-k} \tag{8}$$

To be convinced, it suffices to show that equality (1) is fulfilled for the representation (8). Indeed,

$$\beta X_{n-1} + \alpha\xi_n = \beta\left(\beta^{n-1} X_0 + \sum_{k=0}^{n-2}\alpha\beta^k \xi_{n-1-k}\right) + \alpha\xi_n = \beta^n X_0 + \sum_{k=0}^{n-2}\alpha\beta^{k+1}\xi_{n-(k+1)} + \alpha\xi_n =$$

$$\beta^n X_0 + \sum_{m=1}^{n-1}\alpha\beta^m \xi_{n-m} + \alpha\xi_n = \beta^n X_0 + \sum_{m=0}^{n-1}\alpha\beta^m\xi_{n-m} = X_n$$

By virtue of assumptions made with respect to random variables X_0, α, β and ξ_n from (8) we obtain:

$$E|X_n| \le E|\beta|^m E|X_0| + \sum_{m=0}^{n-1} E|\alpha| E|\beta|^m E|\xi_{n-m}| \le$$

$$\le E|X_0| + E|\alpha|E|\xi_1|\sum_{m=0}^{\infty} E|\beta|^m = E|X_0| + E|\alpha|E|\xi_1|E\left(\sum_{m=0}^{\infty}|\beta|^m\right) =$$

$$= E|X_0| + CE\left(\frac{1}{1-|\beta|}\right) < E|X_0| + \frac{C}{\varepsilon} = K < \infty$$

for all, $n \ge 1$ where $c = E|\alpha|E|\xi_1|$. Then it follows from the Markov inequality that

$$P(|X_n| > n\varepsilon) < \frac{E|X_0|}{n\varepsilon} < \frac{K}{n\varepsilon}$$

Hence it follows that

$$\frac{X_n}{n} \xrightarrow{P} 0 \ as \ n \to \infty \tag{9}$$

(6) follows from (4) and (9).

According to the strong law of large numbers, we have

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k \xrightarrow{a.s.} m = E\xi_1 \ as \ n \to \infty \tag{10}$$

Now, statement 1) of Theorem 1 follows from (5), (6) and (10)

Statement 2) follows from (5) and (10), since the convergence $\frac{X_0}{n} \xrightarrow{a.s} 0, n \to \infty$ is fulfilled.

Theorem 1 is proved.

Theorem 2. Let $\sigma^2 = D\xi_1 < \infty$ and the conditions of theorem 1 be fulfilled Then

$$\lim_{n\to\infty} P\left(\frac{\sqrt{n}}{\sigma}\left(\frac{(1-\beta)}{\alpha}\frac{S_n}{n} - m\right) \le x\right) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \ x \in R.$$

Proof. From (5) we have

$$\frac{1-\beta}{\alpha}\frac{S_n}{n} = \frac{\beta}{\alpha}(X_0 - X_n) + \frac{1}{n}\sum_{k=1}^{n}\xi_k$$

or

$$\frac{1-\beta}{\alpha}\frac{S_n}{n} - m = \frac{\beta}{\alpha}(X_0 - X_n) + \frac{1}{n}\left(\sum_{k=1}^{n}\xi_k - m\right).$$

Multiplying the both hand sides of this equality by $\frac{\sqrt{n}}{\sigma}$ we obtain

$$\frac{\sqrt{n}}{\sigma}\left(\frac{(1-\beta)}{\alpha}\frac{S_n}{n} - m\right) = \frac{\beta}{\alpha\sigma}\frac{X_0 - X_n}{\sqrt{n}} + \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{k=1}^{n}\xi_k - m\right). \tag{11}$$

Repeating the arguments carried out in the proof of theorem 1, it is easy to be convinced that the first term in the right hand side of the equality (11) converges in probability to zero, i.e.

$$\frac{X_0 - X_n}{\sqrt{n}} \xrightarrow{P} 0 \quad as \quad n \to \infty \tag{12}$$

According to the central limit theorem we have

$$\lim_{n\to\infty} P\left(\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{k=1}^{n}\xi_k - m\right) \le x\right) = \Phi(x). \qquad (13)$$

Thus, from (10),(12) and (13) we obtain statement 1 of theorem 2.

From theorem 2 we have

Corollary. Let the conditions of the theorem be fulfilled and let α and β be fixed numbers, $\alpha \neq 0$ and $|\beta| < 1$.

Then

$$\lim_{n\to\infty} P\left(\frac{(1-\beta)S_n - \alpha mn}{\sigma\alpha\sqrt{n}} \le x\right) = \Phi(x), \quad x \in R$$

Hence, in particular, for $\beta = 0$ and $\alpha = 1$ we obtain.

$$\lim_{n\to\infty} P\left(\frac{S_n - nm}{\sigma\sqrt{n}} \le X\right) = \Phi(x).$$

This is a classic central limit theorem for random variables ξ_n, $n \ge 1$.

3. Conclusion.

In the current work, linear boundary value problems for a class of Markov random walks described by a first-order autoregressive process with random coefficients are considered.

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